an important quantity unchanged: the sum $S$ for the Gauss sum or the central temperature for the pentagon problem.

The moral of this analysis is therefore the following: *When there is change, look for what does not change.* These unchanging quantities are known as invariants; they are very likely to be important features of a problem. Then search for symmetries: operations that preserve these invariants.

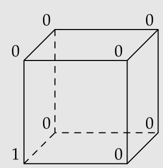
This philosophy of ignoring the changing froth and instead focusing on invariants—the underlying order—underlies the next two tools for discarding apparent complexity: proportional reasoning (Section 1.2.2) and dimensions (Section 1.2.3).

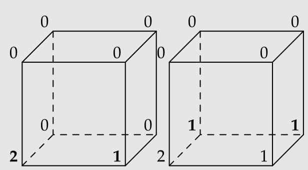
**Problem 1.5 Tiling a termite-eaten chessboard**

Termites found your wooden chessboard and ate away the lower left square and the upper rightmost square. You have an unlimited set of $2 \times 1$ dominoes. Can you use them to tile the modified chessboard? That is, can you lay the dominoes, without any overlap, to cover the whole modified board without extending beyond the board?

**Problem 1.6 Cube solitaire**

A cube starts in the configuration shown in the margin; the goal is to make all eight vertices be multiples of 3 simultaneously. The possible moves are of the form: Pick any edge and increment its two vertices by 1. For example, if you pick the bottom edge of the front face, then the bottom edge of the back face, the configuration becomes the first one in this series, then the second one:





Alas, neither configuration wins the game. Can the game be won? If so, give a sequence of moves ending with all vertices at multiples of 3. If it cannot be won, explain why no move sequence works.



**I like this phrase. I feel it is useful in other areas as well.**

**I like this definition.**

**Given this in our toolbox, we can also look for ways to simplify and slightly modify problems to make a symmetrical model, then we can apply this.**

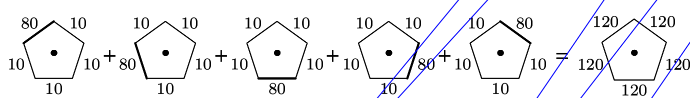
**I found a very easy way of doing it, but I don't think my method incorporates abstraction or estimation. Would someone be willing to share their method so to check if I'm doing something wrong?**

temperatures that lie on top of one another to produce the temperature profile of a new sheet.

*Is adding temperature a legitimate operation?*

Adding temperature is not a legitimate operation in physics. For example, it is nonsense to ask for the total temperature of a hot and cold cup of water. (It is legitimate to ask for the total thermal energy of the two cups.) However, this physical restriction is not built into the heat equation. Therefore, if adding temperatures helps us find a solution to the heat equation—as it shortly will—then the solution will be physically valid. The only requirement embedded into the heat equation is that all operations on the temperatures be linear operations, because the heat equation is a linear differential equation. Fortunately, addition is the canonical linear operation. In conclusion, adding temperatures is a legitimate operation here.

In the new, combined sheet, each edge has a temperature of 120 degrees:




Solving the temperature distribution of this new sheet does not require solving the heat equation! Because all the new sheet's edges are pinned at 120 degrees, the new sheet has a uniform temperature of 120 degrees throughout—even though none of the five constituent sheets has a uniform temperature. Look again at the analogy with Gauss's method, where even though each sum had varying terms (from 1 to 100), the combined sum consisted of constants (all 101). Here, because the centers of the five stacked sheets align and the center temperatures are the same on each sheet, each center has temperature 120/5 = 24 degrees.

Now compare the two examples—the Gauss sum and the pentagon temperature—in order to extract their transferable ideas (the useful abstractions). First, both problems seem difficult upon first glance. The Gauss sum contains 100 terms, all different; the pentagon problem seems to require solving a difficult partial differential equation. Second, both problems contain a symmetry operation. In the Gauss sum, the symmetry operation reversed the order of the terms; in the pentagon problem, the symmetry operation rotated the pentagon by 72 degrees. Third, the symmetry operation left



**Ahh! Not in a million years... oh that's clever.**

**I knew it would be symmetry and I didn't even see this coming.**

**My mind is blown!**

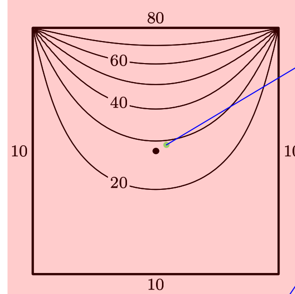
**That's amazing!**

I agree. This trick is genius!

**When I saw the problem, I predicted just taking the average, but then I thought that the non-linear magic of heat would make it not true. Is it symmetry that implies linearity?**

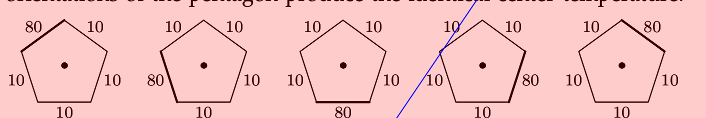
**How do you know when the Gauss sum can be used? (in general)**

$$\nabla^2 T = 0. \tag{1.22}$$

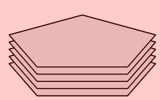



However, even this simpler equation with no time dependence has few easy solutions. And these solutions are not even that simple. For example, on the square piece of foil with edge temperatures as prescribed in the figure (10, 10, 10, and 80 degrees), the temperature distribution is highly nonintuitive. The contour lines, spaced every 10 degrees, are a hard-to-predict shape. The center is surrounded by the 20 and 30-degree contour lines, so the central temperature is somewhere between 20 and 30 degrees. But the exact value is hardly obvious – even for such a regular shape. For a pentagon, even for a regular pentagon, the full temperature distribution is even less intuitive.

Symmetry, however, makes the solution flow. The pentagon is regular, so it looks the same when it is rotated about the center by one-fifth of a circle (an angle of 72 degrees). The only effects of the rotation are to rotate the temperature labels on the edges (the 80-degree edge moves along by one edge) and to rotate the entire temperature distribution. However, the temperature at the center does not change. Therefore, the following five orientations of the pentagon produce the identical center temperature:




Now borrow Gauss's idea of adding the two equivalent ways of summing the series:



$$\begin{aligned} &1 + 2 + 3 + \cdots + 100 \\ &+100 + 99 + 98 + \cdots + 1 \\ \hline &= 101 + 101 + \cdots + 101. \end{aligned} \tag{1.23}$$

Analogously, stack the five equivalent penatgons (equivalent in the sense of having the same central temperature); then, at each spot, add the five

## Comments on page 6

**I'd like to see the contour lines on the pentagram - I don't think it would make the answer too obvious!**

**I haven't done much thermo but it looks like this same symmetricalization could work with E&M when looking at electric and magnetic field due to point charges spaced symmetrically.**

or would this not work because electric fields are vector fields? Can someone answer me?

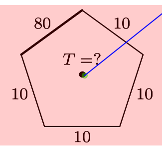
$$\begin{aligned} &1 + 2 + 3 + \cdots + 100 \\ +&100 + 99 + 98 + \cdots + 1 \\ \hline =&101 + 101 + \cdots + 101. \end{aligned} \quad (1.19)$$

In this form, the doubled sum $2S$ consists of 100 copies of 101. So $2S = 100 \times 101$, wherefore

$$S = \frac{1}{2} \times 100 \times 101 = 5050. \quad (1.20)$$

The tedium and complexity of the sum vanished by finding a symmetry operation or, more compactly, a symmetry: a transformation or change that preserves important features of the problem. In the Gauss sum, the symmetry operation is reversing the order of the terms; and the important feature is the total $S$, unchanged by permuting the individual terms.

Symmetry helps simplify not only mathematical problems. For a physical application, imagine a uniform sheet of aluminum foil cut into the shape of a regular pentagon. To its edges attach heat sources holding the edges at the marked temperatures.




*When the temperature distribution stops changing ('comes to equilibrium'), what is the temperature at the center of the pentagon?*

First examine the direct but difficult approach. The temperature on the sheet is a solution of the heat equation, which is the following second-order partial-differential equation:

$$\kappa \nabla^2 T = \frac{\partial T}{\partial t},$$

where $\nabla^2$ is the Laplacian operator, which is the two-dimensional analog of the second derivative; $T$ is the temperature, which is a function of position and time; and $\kappa$ is the material's thermal diffusivity, which is the ratio of the thermal conductivity to the volumetric heat capacity (Section 4.4). For aluminum,

$$\kappa \sim 10^{-4} \frac{\text{meters}^2}{\text{second}} \quad (1.21)$$

After a long-enough time, the temperature distribution settles down and stops changing, so the time derivative $\partial T/\partial t$ approaches zero. The right side of the heat equation vanishes, and the equation simplifies to

## Comments on page 5

**I did it a bit different, but incorporating a similar concept. I saw that adding the two opposite ends = 101. Then the next pair of numbers on the opposite end, and so on. Since I'm dividing the numbers into two groups, a front end and a back end, I realized I'd have 50 101's by the end of it, thereby getting the result - 5050.**

**Awesome. and great example**

**What about thinking about the temperature in the middle as the average of the temperature of the outside?**

**I was thinking of temperature as thermal energy, so the temperature of the middle would be the average of the thermal energy of the sides.**

**80 + 10+10+10+10 = 120.**

**120/5 = 24**

If I didn't know better I would have said this is a better method, but if you think about it, it isn't originally apparent that the temperature in the middle is the average of all temps. Sanjoy does a really great job explaining why this problem is complicated if you choose this method.

**Problem 1.4 Coin-flip game**
Two people take turns flipping a fair coin. Whoever first turns over heads wins. What is the probability that the first player wins?

## 1.2 Discarding fake complexity

Section 1.1 introduced two tools for organizing complexity: divide-and-conquer reasoning and making abstractions. Ater splitting problems into their simplest subproblems or finding reusable ideas, the subproblems may still be too difficult. Then, discard complexity! When this complexity is only apparent, discarding it simplifies the problem without losing information. Three tools for discarding apparent complexity are symmetry and conservation (Section 1.2.1), proportional reasoning (Section 1.2.2), and dimensional analysis (Section 1.2.3).

### 1.2.1 Symmetry and conservation

Symmetry beautifully simplifies any problem to which it applies—without sacrificing any accuracy. A classic example is the following story about the brilliant mathematician Carl Friedrich Gauss (1777–1855). The story may be mere legend, but it is so instructive for our purposes that it ought to be true. One day, when Carl Friedrich was in primary school, the story goes, his schoolteacher was angry at the students and wanted to occupy them and obtain thereby some peace. The teacher asked the students to compute the sum

$$S = 1 + 2 + 3 + \cdots + 100, \quad (1.17)$$

and sat back to enjoy a welcome break. To the teacher's surprise, Gauss returned in a few minutes claiming that the sum is 5050.

*Was Gauss right? If so, how can the sum be computed so quickly?*

Gauss noticed that the sum remains unchanged when the terms are added in the opposite order, from 100 down to 1. In other words,

$$100 + 99 + 98 + \cdots + 1 = 1 + 2 + 3 + \cdots + 100. \quad (1.18)$$

Then Gauss added the two alternative but equal sums:



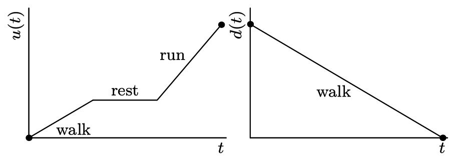
**What does "only apparent" mean? Only in your mind?**

Meaning non-essential to the problem.

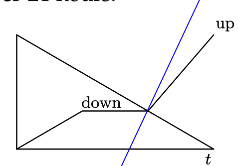
**I see. the complexity is "only in your head" here because of the way the problem is phrased. if it were phrased 101*100 / 2 then it would be simpler.**

*Must $u(t) = d(t)$ at some time $t$? Or can you choose $u(t)$ and $d(t)$ so that $u(t)$ is never equal to $d(t)$?*

These abstractions make the question clean but not clean enough to answer at a glance. The goal of answering the question at a glance suggests making a visual representation or diagram. Here are diagrams illustrating possible upward and downward schedules.




The upward schedule has three 8 hours segments. In the first segment, with its gentle slope, you stroll up the mountain; in the second, flat segment, you take an 8-hour nap. Energized by the nap, in the third segment, with its steep slope, you run for 8 hours and reach the top. On the way down, you walk steadily down the mountain over 24 hours.




Now draw these upward and downward schedules on the same diagram. The paths intersect! The intersection point gives the time and location where the upward and downward schedules landed on the same point at the same time of day (but on separate days). Furthermore, the diagram shows that this pattern is general. No matter the schedule, the upward path creates a path that the downward path must cross. At their intersection, the position along the upward and downward paths is identical as is the time of day. Therefore, there is always a point that you reached at the same time of day going up and down – a conclusion hard to reach without abstracting away the unessential details to make a diagram.

The tool of making abstractions will reappear in many guises.

**Problem 1.3 What details are relevant?**
If you rest at the top for 48 hours instead of 24 hours, does the conclusion change? What if you rest for 12 hours?

# Comments on page 3

**Is there a way to do this with partial integrals that would be convenient? If I found a more general method, that would be a greater degree of abstraction right?**

**Is there a way to visualize other types of problems that don't involve position or any other equation of motion with diagrams?**

The first example (with the series) and problem 1.4 are both good examples of visualizing with numbers.

I really want to be able to solve this numerically without the graph.

You can. Call v(t) = u(t) - d(t), and say that h is the total height of the mountain. At t = 0, v = -h and at the end time, v = h. V is continuous, so by the intermediate value theorem V must be 0 at some point, which means that u(t) = d(t).

But the picture is so much prettier.

$$1 + \frac{2}{3} + \frac{4}{9} + \cdots, \tag{1.13}$$

where each term has the form $(2/3)^n$. With slightly blurry vision, the entire series except for the first term looks similar to the entire series:

$$\frac{2}{3} + \frac{4}{9} + \cdots = \frac{2}{3} \times \left(1 + \frac{2}{3} + \cdots\right). \tag{1.14}$$

The series in parenthesis is the original series! Therefore, the original series is a reusable module. Give it a name; say, $S$. Then

$$S = 1 + \frac{2}{3}S. \tag{1.15}$$

With this abstraction, the sum becomes easy to find:

$$S = 3. \tag{1.16}$$

For further practice making abstractions, see what remains in the following problem after ignoring the froth.

> Starting on June 5th at noon, you hike along a path all the way up Mount Fuji over a 24-hour period, resting along the way as you need. At the top, you sleep for 24 hours. Starting on June 7th at noon, you walk all the way down the same path over the following 24 hours. Were you at any point on the path at the same time of day on the way up and on the way down (using a 24-hour clock for time of day)? Alternatively, is it possible to walk up and down on a careful schedule ensuring that there is no such point?

First find the details that definitely do not matter for answering the question. These include the name of the mountain, its height, the length of the path, the month, and the day of the month. All that matters is the schedule on which you walk up and down: where you are at what time of day. A schedule can be represented as a function giving your position on the path as a function of time. Because the date does not matter, only the time of day (on a 24-hour clock), the time $t$ would run from 0 to 24 hours. For position on the path, use the range 0 to 1, where 0 means the bottom of the mountain, and 1 means the top of the mountain. That range suggests a further simplification: Specify the time from 0 to 1 instead of 0 to 24 hours.

A journey consists of two schedules: $u(t)$ for hiking up the mountain and $d(t)$ for hiking down the mountain. In this abstract representation, the problem becomes the following.



**I want to practice this more. It seems really useful.**

**This method can be also be used to elegantly solve a general cases for infinite geometric series. S = r + r^2 + r^3 S / r = 1 + S S = r + Sr 0 = r + S(r-1) S = -r/(r-1)**

**S - (2/3)S= (1/3)S = 1, so S = 3**

**I didn't realize it was so simple at first.**

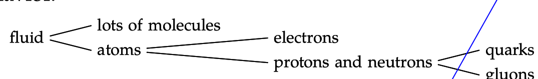
**Ha! My brother put this question to me, and I soon saw the way to the answer, but I tried to use it to prove that you 'missed' yourself! It did help that in his telling, all the extraneous detail was already removed**

### 1.1.2 Abstraction

In the oil-imports problem, the final tree with numbers compactly represents the estimate. The tree representation is an example of an abstraction – the second tool for organizing complexity. An abstraction's essential characteristic is reusability. As explained in software design [1, Section 1.1.8]:

> The importance of this decomposition strategy [abstraction] is not simply that one is dividing the program into parts. After all, we could take any large program and divide it into parts – the first ten lines, the next ten lines, the next ten lines, and so on. Rather, it is crucial that each procedure accomplishes an identifiable task that can be used as a module in defining other procedures.

This principle applies generally: Find units of thought that can be reused in understanding and building other systems. The tree representation for divide-and-conquer estimates is a reusable unit of thought. Another, more familiar unit, is the idea of a fluid. It is our own mental construction – and a powerful, reusable one. The behavior of water is far removed from the actors of fundamental physics: quarks and electrons. But quarks combine to build protons and neutrons. Protons, neutrons, and electrons combine to build atoms. Atoms combine to build molecules. And molecules, in large collections, behave in such repeatable ways that we collect the properties together into a single group and give it a name: fluid behavior.




The idea of a fluid becomes a new unit of thought that helps explain diverse phenomena, without our having to calculate anew or even to know how quarks and electrons eventually produce fluid behavior.

The preceding definition described an abstraction by the constraint that an abstraction must be reusable. That constraint helps us see how to make reusable units of thought. Every situation, if specified in full detail, is unique. Therefore, a full description cannot form an abstraction. Instead, look at a situation with slightly blurry vision and ignore the froth; what remains is an abstraction.

To warm up with abstraction, we first find the following infinite sum:

**To be completely honest, I feel that we've already been doing this. We take the situation, break it down to several general ideas that helps us simplify our calculations. In fact, I almost feel that divide and conquer cannot be done without undergoing this process first. I'd like to hear other people's opinions on this.**

Yes, I think it is repetitive. Maybe reiterating this main point is important so we remember it in the long run?

I think the main takeaway is that the idea is reusable, which is not necessarily true of divide and conquer.

I actually think that this isn't really saying anything about actually breaking down the situation into simple means to divide and conquer. I think this particular sentence is talking about not narrowing down. I know it sounds the same. but I mean, being able to keep the situation broad to include all possible branches instead of simplifying (though i agree that simplifying is crucial in the end.)

You do seem to make a point until that last bit, where you talk about including all possible branches. To me, the major point is quite the opposite - ignore some of those branches to simplify the situation, the idea which I was originally referring to.

**It's interesting that you can get a solution with greater accuracy by eliminating information.**

**confused how we went from abstraction and branching straight into an infinite sum. what are we trying to see here?**

You probably noticed by the time you reached the next page, but it seems like it was just an example of simple abstraction that we can do, hence "To warm up with abstraction..."

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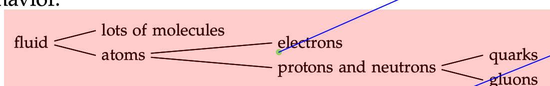


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To warm up with abstraction, we first find the following infinite sum:

## Comments on page 1

**Here's the second reading! It introduces two new tools. Make your comments by Tuesday (2/8) at noon.**

**For the homework, try problems 1.3-1.6 that are in the text.**

**So less about dividing, more about breaking down.**

where "breaking down" is defined as 'usefully and selectively separating into components'

**So, the software analogy here would be classes, methods, etc. The more often a block of code is used, the more likely it is to be divided into its own 'unit of thought.'**

**You can plug one tree into the limb of another tree**

**I think the interesting take-home here is the importance of working the problem at the appropriate level of abstraction. When you focus at too specific of a depth, the problem will take too long and probably be too complex to do correctly anyways. At too high of a level, you miss details that make a meaningful difference.**

**I'm learning about nonlinear dynamics right now, and by definition, the law of superposition would not work on dynamical problems. How would you estimate for a chaotic problem? I suppose the only way you could estimate was if instead of looking at localized pieces, you attempted to estimate the big picture using symmetry maybe?**